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ON GENERAL PROBLEMS WITH HIGHER

DERIVATIVE BOUNDED STATE VARIABLES

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This paper is a sequel to an article by the author, concerned with a canonical control problem involving state constraints in which the control enters in the second derivative of the constraint. Extensions of the results obtained there are developed herein for a general form of the control problem of Bolza with the above type of constraints. It is also shown that modified forms hold true for the relation $\dot{H} = H_t$ and for the transversality relation usually obtained in problems of this type.

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1. Introduction.

The present paper serves as the sequel to Ref. 2 in extending the results obtained there to the following general problem in optimal control: Let C be class of arcs

$$a: \quad x^i(t), \quad \dot{x}^i(t), \quad u^k(t), \quad b^\sigma \quad t^0 \le t \le t^1$$
$$i=1,\dots,N \qquad k=1,\dots,K \qquad \sigma=1,\dots,r$$

whose points $(t,x(t),\dot{x}(t),u(t))$ lie in a region R in t-x-$\dot{x}$-u space, with b in a region B in b space, u(t) piecewise continuous, x(t) of class C^1 and which in addition satisfy the constraints:

$$\ddot{x}^i = f^i(t,x,\dot{x},u), \qquad i=1,\dots,N$$

$$\psi^\alpha(t,x) \le 0, \quad 1 \le \alpha \le m', \quad \psi^\alpha(t,x) = 0, \quad m' < \alpha \le m$$

$$\theta^\eta(t,x,\dot{x},u) \le 0, \quad 1 \le \eta \le L', \quad \theta^\eta(t,x,\dot{x},u) = 0, \quad L' < \eta \le L$$

$$I_\gamma(a) \le 0, \quad 1 \le \gamma \le p', \quad I_\gamma(a) = 0, \quad p' < \gamma \le p$$

$$x^i(t^s) = X^{is}(b), \quad \dot{x}^i(t^s) = \dot{X}^{is}(b), \quad t^s = T^s(b), \quad s = 0,1^{(1)}$$

where

$$I_\gamma(a) = g_\gamma(b) + \int_{t^0}^{t^1} L_\gamma(t,x,\dot{x},u)dt, \qquad \gamma = 1,\dots,p .$$

It is desired to minimize an integral of the form

$$I_o(a) = g_o(b) + \int_{t^0}^{t^1} L_o(t,x,\dot{x},u)dt$$

on the class C .

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(1) With few exceptions, superscript • will denote differentiation with respect to t . The exceptions are for notational convenience and will be explicitly noted. The current exception is the term $\dot{X}^{is}$ and refers to the constraints on $\dot{x}^i(t)^s$. Also unless otherwise specified, the indices i,k,σ,α,η will have the respective ranges:

$$1 \le i \le N, \quad 1 \le k \le K, \quad 1 \le \sigma \le r, \quad 1 \le \alpha \le m, \quad 1 \le \eta \le L.$$

In this paper, modified forms of the relation $dH/dt = H_t$ and of the transversality relation usually obtained in problems of this type are shown to hold.

The result for the above stated general problem is Theorem 9.1 and is proven by considering a sequence of problems each with more of the special restrictions which were present in Ref. 2 now deleted. Corresponding to Theorem 3.2 of Ref. 2, we will prove a set of necessary conditions for each problem. In the course of these proofs, the unproven statements involving (11) and (19) respectively of Theorem 3.1 and 3.2 of Ref. 2 will be proven. It is noted that unless otherwise stated, the conventions of Ref. 2 apply also to this paper.

2. A Problem Over a Variable Time Interval.

As a first step in generalizing our problem, we allow the arcs under consideration to be over a variable time interval. We are now concerned with arcs

$$a: \qquad x^i(t), \qquad \dot{x}^i(t), \qquad u^k(t), \qquad b^\sigma, \qquad t^0 \leq t \leq t^1$$

$$1 \leq i \leq N, \qquad 1 \leq k \leq K \qquad 1 \leq \sigma \leq r$$

(with the interval $[t^0, t^1]$ depending on a) which have points $(t,x(t),\dot{x}(t),u(t))$ in a region R in t-x-$\dot{x}$-u space, b in a region B in b space, $u(t)$ piecewise continuous, and $x(t)$ of class C^1.

We desire to minimize an integral of the form

$$I_0(a) = g_0(b) + \int_{t^0}^{t^1} L_0(t,x(t),\dot{x}(t),u(t))dt \tag{1}$$

on the class C of arcs which satisfy:

$$\ddot{x}^i = f^i(t,x(t),\dot{x}(t),u(t)) \qquad 1 \leq i \leq N \tag{2-}$$

$$\psi^\alpha(t,x(t)) \leq 0 \qquad 1 \leq \alpha \leq m \tag{2-}$$

$$I_\gamma(a) \leq 0, \quad 1 \leq \gamma \leq p', \quad I_\gamma(a) = 0, \quad p' < \gamma \leq p \tag{2-}$$

$$x^i(t^s)=X^{is}(b), \quad \dot{x}^i(t^s)=\dot{X}^{is}(b), \quad t^s = T^s(b), \quad s = 0,1 \tag{2-}$$

where:

$$I_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} L_\gamma(t,x(t),\dot{x}(t),u(t))dt \qquad \gamma = 1,\dots,p .$$

It will be assumed that the functions f^i, L_γ, g_γ, X^{is}, $\dot{X}^{is}$, T^s, are of class C^1 on R or B as the case may be, while the functions ψ^α are of class C^3.

Analogous to [2], let

$$\tilde{\phi}^\alpha(t,x,\dot{x}) = \psi^\alpha_t + \psi^\alpha_{x^i}\dot{x}^i \tag{3}$$

$$\phi^\alpha(t,x,\dot{x},u) = \psi^\alpha_{t^2} + 2\psi^\alpha_{tx^i}\dot{x}^i + \psi^\alpha_{x^ix^j}\dot{x}^i\dot{x}^j + \psi^\alpha_{x^i} f^i .$$

For arcs that satisfy (2-1) these functions act as $d\psi^\alpha/dt$ and $d^2\psi^\alpha/dt^2$ along these arcs. Furthermore according to the properties of ψ^α, and f^i, these functions are respectively of class C^2 and C^1 on R.

Let R_o be the set of points $(t,x,\dot{x},u)$ in R satisfying the conditions

$$\psi^\alpha \leq 0 \tag{4}$$

and for all α with $\psi^\alpha = 0$, then $\tilde{\phi}^\alpha = 0$ and $\phi^\alpha \leqq 0$

or for all α with $\psi^\alpha = 0$, then $\tilde{\phi}^\alpha = 0$ and $\phi^\alpha \geqq 0$.

We desire to test whether a given arc

$$a_o: \qquad x_o(t), \quad \dot{x}_o(t), \quad u_o(t), \quad b_o \qquad t^o_o \leq t \leq t^1_o$$

which is in the class C, is a solution to the above problem. Define the set R_1 as the collection of points $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. It will be assumed that the matrix

$$\begin{bmatrix} \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \alpha,\beta = 1,\dots,m \qquad k = 1,\dots,K \tag{5}$$

[where $\delta_{\alpha\beta}$ denotes the Kronecker Delta] has rank m on R_1.

3. First-Order Necessary Conditions for a Minimum.

Define the functions[2]

$$H(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu}) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}_\alpha \tilde{\phi}^\alpha$$

$$G(b) = \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{io} + \lambda_{p+N+1} T^o + \lambda_{\bar{p}+i} \dot{X}^{io}$$

$$- \lambda_{\bar{\bar{p}}+\alpha}\left(\dot{q}_i^\alpha(t_o^1)X^{i1} + \dot{q}_t^\alpha(t_o^1)T^1 + q_i^\alpha(t_o^1)\dot{X}^{i1}\right) + \lambda_{\bar{\bar{p}}+m+\alpha}\left(\dot{q}_i^\alpha(t_o^o)X^{io} + \dot{q}_t^\alpha(t_o^o)T^o + q_i^\alpha(t_o^o \ldots\right.$$

$$1 \leq \gamma \leq p; \qquad 1 \leq i \leq N; \qquad 1 \leq \alpha \leq m; \qquad \bar{p} = p+N+1; \qquad \bar{\bar{p}} = p+2N+1 .$$

With these definitions, the result to be proven for the present problem is the following theorem:

Theorem 3.1. Suppose that the arc a_o is a solution to the above problem Then there are multipliers:

$$K^\tau, \; \lambda_\rho \; , \; \mu_\alpha(t), \; \tilde{\mu}_\alpha \; , \; p_i(t), \; \tilde{p}_i(t)$$

$$\tau = 1,\ldots,2m, \quad \rho = 0,1,\ldots p+2N+1+2m, \quad \alpha = 1,\ldots,m, \quad i = 1,\ldots,N$$

and functions H, G as described above such that with these multipliers as arguments then the following conditions hold:

The inequality

$$\begin{aligned} &H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \\ &\qquad \leq H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \end{aligned} \tag{6}$$

is valid for all u with $(t,x_o(t),\dot{x}_o(t),u)$ in R_o . The multipliers $p_i(t)$, $\tilde{p}_i(t)$ are each continuous on $[t_o^o \, , \, t_o^1]$ and together with $\mu(t),\tilde{\mu},\lambda$ satisfy the relations:

$$\dot{p}_i = -H_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -H_{x^i} \qquad \ddot{x}^i = H_{p_i} \qquad H_{u^k} = 0 \tag{7}$$

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(2) Analogous to [2] the symbols $q_i^\alpha(t)$, $q_t^\alpha(t)$ respectively denote $\psi^\alpha_{x^i}(t,x_o(t))$, $\psi_t^\alpha(t,x_o(t))$. In addition, functions $M(t,x,\dot{x},u)$ when evaluated along a_o at points $(t,x_o(t),\dot{x}_o(t),u_o(t))$ will often be referred to as $M(t)$.

along a_o on intervals of continuity of $u_o(t)$. The function $H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu})$ is continuous along a_o. On intervals of continuity of $u_o(t)$, this function has a continuous derivative satisfying[3]

$$dH/dt + \dot{\mu}_\alpha \phi^\alpha = H_t \quad . \tag{8}$$

The transversality condition[3]

$$dG + \left[\left(-H(t_o^s) - \dot{\mu}_\alpha(t_o^s)\tilde{\phi}^\alpha(t_o^s)\right)dT^s + p_i(t_o^s)d\dot{X}^{is} + \tilde{p}^i(t_o^s)dX^{is}\right]_{s=0}^{s=1} = 0 \tag{9}$$

is valid along a_o for all db. The multipliers λ_ρ, K^α are constants which satisfy:

$$\begin{aligned}
&\lambda_o \geq 0 \quad \lambda_\gamma \geq 0 \quad \text{with} \quad \lambda_\gamma = 0 \quad \text{if} \quad I_\gamma(a_o) < 0 \qquad 1 \leq \gamma \leq p' \\
&K^\alpha \geq 0 \qquad \lambda_{p+i} = K^\alpha q_i^\alpha(t_o^o) + K^{m+\alpha}\dot{q}_i^\alpha(t_o^o) \\
&\lambda_{p+N+1} = K^\alpha q_t^\alpha(t_o^o) + K^{m+\alpha}\dot{q}_t^\alpha(t_o^o) \qquad \lambda_{p+N+1+i} = K^{m+\alpha} q_i^\alpha(t_o^o) \; .
\end{aligned} \tag{10}$$

Furthermore, together with $\mu(t),\tilde{\mu},p(t),\tilde{p}(t)$ they are not of the form[3]

$$\begin{aligned}
&\lambda_\gamma = 0 \quad \gamma = 0,1,\ldots,p, \quad K^\tau = 0 \quad \tau = 1,\ldots,2m, \\
&\dot{\mu}_\alpha(\bar{t}) = a_\alpha \quad \text{if} \quad \psi^\alpha(\bar{t}) < 0, \quad \mu_\alpha(\bar{t}) = a_\alpha \bar{t} + b_\alpha , \\
&\tilde{\mu}_\alpha = a_\alpha , \; \lambda_{p+2N+1+\alpha} = a_\alpha t_o^1 + b_\alpha , \; \lambda_{p+2N+1+m+\alpha} = a_\alpha t_o^o + b_\alpha , \\
&p_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha)q_i^\alpha(\bar{t}) , \qquad \tilde{p}_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha)\dot{q}_i^\alpha(\bar{t})
\end{aligned} \tag{11}$$

for any constants a_α, b_α at any point $\bar{t}$ in $[t_o^o , t_o^1]$.

For each α the multiplier $\mu_\alpha(t)$ is continuous on intervals of continuity of $u_o(t)$ and satisfies the following properties: (i) there are constants a_α, b_α such that $\mu_\alpha(t)-(a_\alpha t + b_\alpha)$ is a nonincreasing function on $[t_o^o,t_o^1]$; (ii) it is of the form $\bar{a}_\alpha t + \bar{b}_\alpha$ on intervals upon which $\psi^\alpha(t)<0$ and (iii) $\dot{\mu}_\alpha(t_o^1) = \tilde{\mu}_\alpha$ if $\psi^\alpha(t_o^1) < 0$.

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(3) In the process of proving (8), (9) the terms involving $\dot{\mu}_\alpha$ will be shown to exist.

4. Transformation of the Problem.

In order to prove Theorem 3.1, introduce the additional variables, y, ν, b^{r+1}, b^{r+2} and the condition

$$y'' = \nu \ . \qquad (12)$$

The prime denotes differentiation with respect to a parameter s and all elements will be represented with respect to s, where s ranges on the fixed interval $[t_o^o, t_o^1]$ associated with the arc a_o .

Define $\bar{B} = B \times B^{r+1} \times B^{r+2}$ (where B is the space of the original problem and B^{r+1}, B^{r+2} are the spaces of the variables b^{r+1}, b^{r+2}) and $\bar{R}$ as the set of points (s,x,x',y,y',u,ν) with $y' > 0$, $y = t$, $x' = \dot{x}y'$ and $(t,x,\dot{x},u)$ in R. The functions $f^i,\psi^\alpha,\phi^\alpha,\tilde{\phi}^\alpha,L_\gamma$ will be considered to be defined both on $\bar{R}$ and on R as follows: When in $\bar{R}$, the variables y, x'/y' respectively will assume the roles of the variables $t,\dot{x}$ as arguments for these functions, while when in R these functions will depend upon $t,\dot{x}$ in the usual manner.

We are now concerned with arcs $\bar{a}$ in $\bar{R}$ of the form:

$$\bar{a}: \qquad \bar{x}(s), \quad \bar{x}'(s), \quad \bar{y}(s), \quad \bar{y}'(s), \quad \bar{u}(s), \quad \bar{\nu}(s), \quad \bar{b} \qquad t_o^o \le s \le t_o^1$$

where $\bar{b}$ has been augmented with the variables b^{r+1}, b^{r+2} introduced above.

Consider the following problem: It is desired to minimize the integral

$$\bar{I}_o(\bar{a}) = g_o(\bar{b}) + \int_{t_o^o}^{t_o^1} \bar{L}_o(\bar{x}(s),\bar{x}'(s),\bar{y}(s),\bar{y}'(s),\bar{u}(s),\bar{\nu}(s))ds$$

with $\bar{L}_o \equiv L_o y'$ on the class $\bar{C}$ of arcs $\bar{a}$ that have points $(s,\bar{x}(s),\bar{x}'(s),\bar{y}(s),\bar{y}'(s),\bar{u}(s),\bar{\nu}(s))$ in $\bar{R}$, $\bar{b}$ in $\bar{B}$, $\bar{u}(s)$, $\bar{\nu}(s)$ piecewise continuous, $\bar{x}(s)$ of class C^1 and in addition satisfy the conditions:

$$\bar{x}''^i = \bar{f}^i = f^i[\bar{y}']^2 + \frac{\bar{x}'^i}{\bar{y}'}\bar{\nu} \qquad \bar{y}'' = \bar{\nu} \qquad (13$$

$$\psi^\alpha \le 0 \qquad (13$$

$$\bar{I}_\gamma(\bar{a}) \leq 0 \qquad 1 \leq \gamma \leq p' \qquad \bar{I}_\gamma(\bar{a}) = 0 \qquad p' < \gamma \leq p \tag{13-3}$$

$$\begin{aligned} &\bar{x}^i(t_o^s) = X^{is}(\bar{b}) \qquad &&\bar{x}^{i\prime}(t_o^s) = \dot{X}^{is}(\bar{b})T'^s(\bar{b}) \\ &\bar{y}(t_o^s) = T^s(\bar{b}) \qquad &&\bar{y}'(t_o^s) = T'^s(\bar{b}) \qquad s = 0,1 \end{aligned} \tag{13-4}$$

where: (i) $T'^s(\bar{b}) \equiv b^{r+s+1}$ $s = 0,1$; (ii) prime denotes differentiation with respect to s; (iii) in (13-4) $\dot{X}^{is}$ are the functions of the original problem and finally, with $\bar{L}_\gamma = L_\gamma y'$

$$\bar{I}_\gamma(\bar{a}) = g_\gamma(\bar{b}) + \int_{t_o^o}^{t_o^1} \bar{L}_\gamma ds \qquad \gamma = 1,\ldots,p\,. \tag{14}$$

Set

$$\tilde{\bar{\phi}}^\alpha = \tilde{\phi}^\alpha y' \qquad \bar{\phi}^\alpha = \phi^\alpha[y']^2 + \tilde{\phi}^\alpha \nu \tag{15}$$

and let $\bar{R}_o$ be defined in an analogous manner with respect to ψ^α, $\tilde{\bar{\phi}}^\alpha$, $\bar{\phi}^\alpha$ as was R_o defined with respect to ψ^α, $\tilde{\phi}^\alpha$, ϕ^α. Now given an arc $\bar{a}$ in $\bar{R}$ satisfying the conditions (13), then we can find an arc a in R satisfying (2). By similar reasoning, and by recognizing that we are essentially only changing the variable of integration in speaking of corresponding arcs in $\bar{R}$ and R, then we see that the arc

$$\bar{a}_o: \quad \begin{aligned} &\bar{x}_o(s) = x_o(s) \qquad &&\bar{x}'_o(s) = \dot{x}_o(s) \qquad &&\bar{y}_o(s) = s \\ &\bar{y}'_o(s) = 1 &&\bar{u}_o(s) = u_o(s) &&\bar{\nu}_o(s) = 0 \\ &\bar{b}_o^\sigma = b_o^\sigma \quad 1 \leq \sigma \leq r &&\bar{b}_o^{r+1} = \bar{b}_o^{r+2} = 1 &&t_o^o \leq s \leq t_o^1 \end{aligned} \tag{16}$$

(where $x_o(s)$, $\dot{x}_o(s)$, $u_o(s)$, b_o are from the arc a_o) will be a solution to the transformed problem if a_o is a solution to the original one.

Next let $\bar{R}_1$ be the collection of points $(s,\bar{x}_o(s),\bar{x}'_o(s),\bar{y}_o(s),\bar{y}'_o(s),u,\nu)$ which are in $\bar{R}_o$. By the definition of $\tilde{\bar{\phi}}^\alpha$, $\bar{\phi}^\alpha$, $\bar{a}_o$ we see that if $(\bar{s},\bar{x}_o(\bar{s}),\bar{x}'_o(\bar{s}),\bar{y}_o(\bar{s}),\bar{y}'_o(\bar{s}),\bar{u},\bar{\nu})$ is a point in $\bar{R}_1$ then with $\bar{t} = \bar{y}_o(\bar{s}) = \bar{s}$, $\dot{\bar{x}}_o(\bar{s}) = \bar{x}'_o(\bar{s})$, the point $(\bar{t},\bar{x}_o(\bar{t}),\dot{\bar{x}}_o(\bar{t}),\bar{u})$ is in R_1. Furthermore the following matrix relation is true

$$\left[\bar{\phi}^\alpha_{u^k},\ \bar{\phi}^\alpha_\nu\ ,\ \delta_{\alpha\beta}\psi^\beta\right] = \left[\phi^\alpha_{u^k},\ \tilde{\phi}^\alpha\ ,\ \delta_{\alpha\beta}\psi^\beta\right] \tag{17}$$

where the left-hand and right-hand sides of (17) are evaluated at the respective points introduced above in $\bar{R}_1$ and R_1. Then by the assumption concerning the matrix of (5), we see that the matrix on the left hand side of (17) has rank m on $\bar{R}_1$. Thus the transformed problem is of the type described in Section 2 of Ref. 2.

5. Proof of Theorem 3.1.

Recall that the proofs of the statements involving (11) and (19) of Theorems 3.1 and 3.2 of [2] were deferred. In this section, we shall prove Theorem 3.1 by referring to Theorem 3.2 of [2] but without the use of the statement involving (19) of that theorem. We shall then retrace our steps to establish the statements in question in Theorems 3.1 and 3.2 of [2].

Consider now that Theorem 3.2 of [2] except for (19) and its associated remark is applied to the present problem. Then there are multipliers K^τ, $\bar{\lambda}_\rho$, $\mu_\alpha(s)$, $\tilde{\mu}_\alpha$, $p_i(s)$, $\tilde{p}_i(s)$ $\quad \tau = 1,\dots,2m$, $\quad \rho=0,1,\dots,p+2N+2+2m$, $\quad \alpha=1,\dots,m$, $\quad i=1,\dots,N+1$ and functions $\bar{H}$, $\bar{G}$ which according to Theorem 3.2 of [2] together with the definition of $\bar{f}^i$, $\bar{L}_\gamma$, $\bar{\phi}^\alpha$, $\tilde{\bar{\phi}}^\alpha$ take the form:

$$\begin{aligned}\bar{H}(s,x,y,x',y',u,\nu,p,\tilde{p},\mu,\tilde{\mu}) &= p_i\left[f^i[y']^2 + \frac{x'^i}{y'}\nu\right] + \\ &+ p_{n+1}\nu + \tilde{p}_i\frac{x'^i}{y'}y' + \tilde{p}_{N+1}y' - \bar{\lambda}_o L_o y' - \bar{\lambda}_\gamma L_\gamma y' \\ &- \mu_\alpha[\phi^\alpha[y']^2 + \tilde{\phi}^\alpha\nu] - \tilde{\mu}_\alpha\tilde{\phi}^\alpha y'\end{aligned} \tag{18}$$

(where the arguments of f^i, L_o, L_γ, ϕ^α, $\tilde{\phi}^\alpha$ are $(y,x,\overset{\prime}{x}/\overset{\prime}{y},u)$ as explained below (12) and where we have for convenience multiplied and divided by $\overset{\prime}{y}$ as factor of $\tilde{p}_i$) and

$$\bar{G} = \bar{\lambda}_o g_o + \bar{\lambda}_\gamma g_\gamma + \bar{\lambda}_{p+i} X^{io} + \bar{\lambda}_{p+N+1} T^o + \bar{\lambda}_{\bar{p}+i} \dot{X}^{io}\overset{\prime}{T}^o$$
$$+ \bar{\lambda}_{\bar{p}+N+1}\overset{\prime}{T}^o - \bar{\lambda}_{\hat{p}+\alpha}\left[q_i^\alpha(t_o^1)\dot{X}^{i1}\overset{\prime}{T}^1 + q_y^\alpha(t_o^1)\overset{\prime}{T}^1 + \overset{\prime}{q}{}_i^\alpha(t_o^1)X^{i1} + \overset{\prime}{q}{}_y^\alpha(t_o^1)T^1\right] \tag{19}$$
$$+ \bar{\lambda}_{\hat{p}+m+\alpha}\left[q_i^\alpha(t_o^o)\dot{X}^{io}\overset{\prime}{T}^o + q_y^\alpha(t_o^o)\overset{\prime}{T}^o + \overset{\prime}{q}{}_i^\alpha(t_o^o)X^{io} + \overset{\prime}{q}{}_y^\alpha(t_o^o)T^o\right]$$

$$\bar{p} = p+N+1, \qquad \hat{p} = p+2(N+1), \qquad 1 \le i \le N, \qquad 1 \le \gamma \le p, \qquad 1 \le \alpha \le m$$

(where: (i) the term $\dot{X}^{io}\overset{\prime}{T}^o$ is the product of the functions $\dot{X}^{io}$, $\overset{\prime}{T}^o$ introduced respectively in the original and transformed problems and this product forms the constraint on $\overset{\prime}{x}{}^i(t_o^o)$; (ii) the term $\overset{\prime}{q}{}_y^\alpha$ means $d\psi_y^\alpha(s)/ds$ and (iii) analogous statements hold for the other terms.)

When the above listed multipliers are used as arguments in these functions, then the following conditions hold:

The inequality

$$\bar{H}(s,\bar{x}_o(s),\bar{y}_o(s),\overset{\prime}{\bar{x}}_o(s),\overset{\prime}{\bar{y}}_o(s),u,\nu,p(s),\tilde{p}(s),\mu(s),\tilde{\mu}) \le$$
$$\bar{H}(s,\bar{x}_o(s),\bar{y}_o(s),\overset{\prime}{\bar{x}}_o(s),\overset{\prime}{\bar{y}}_o(s),\bar{u}_o(s),\bar{\nu}_o(s),p(s),\tilde{p}(s),\mu(s),\tilde{\mu}) \tag{20}$$

is valid for all points $(s,\bar{x}_o(s),\bar{y}_o(s),\overset{\prime}{\bar{x}}_o(s),\overset{\prime}{\bar{y}}_o(s),u,\nu)$ in $\bar{R}_o$. The relations

$$\overset{\prime}{p}_i = -\bar{H}_{\overset{\prime}{x}{}^i}, \qquad \overset{\prime}{p}_{N+1} = -\bar{H}_{\overset{\prime}{y}}, \qquad \overset{\prime}{\tilde{p}}_i = -\bar{H}_{x^i}, \qquad \overset{\prime}{\tilde{p}}_{N+1} = -\bar{H}_y$$
$$\overset{\prime\prime}{x}{}^i = \bar{H}_{p_i} \qquad \overset{\prime\prime}{y} = \bar{H}_{p_{N+1}} \qquad \bar{H}_{u^k} = 0 \qquad \bar{H}_\nu = 0 \tag{21}$$

hold along intervals of continuity of $\bar{u}_o(s)$. The transversality condition

$$d\bar{G} + \left[p_i(t_o^s)[d\dot{X}^{is}\overset{'}{T}^s + \dot{X}^{is}d\overset{'}{T}^s] + p_{N+1}(t_o^s)d\overset{'}{T}^s + \tilde{p}_i(t_o^s)dX^{is} + \tilde{p}_{N+1}(t_o^s)dT^s \right]_{s=0}^{s=1} = 0 \tag{22}$$

is valid along $\bar{a}_o$ for all $d\bar{b}$. Furthermore, the multipliers $\bar{\lambda}, K, \mu(s), \tilde{\mu}, p(s), \tilde{p}(s)$ are not of the form:

$$\begin{aligned}
&\bar{\lambda}_\gamma = 0 \quad \gamma = 0,1,\ldots,p, \quad K^\tau = 0 \quad \tau = 1,\ldots,2m, \quad \overset{'}{\mu}_\alpha(\bar{s}) = a_\alpha \quad \text{if} \quad \psi^\alpha(\bar{s}) < 0 \\
&\mu_\alpha(\bar{s}) = a_\alpha\bar{s} + b_\alpha\,, \quad \tilde{\mu}_\alpha = a_\alpha, \quad \bar{\lambda}_{\hat{p}+\alpha} = a_\alpha t_o^1 + b_\alpha, \quad \bar{\lambda}_{\hat{p}+m+\alpha} = a_\alpha t_o^o + b_\alpha \\
&p_i(\bar{s}) = [a_\alpha\bar{s} + b_\alpha]q_i^\alpha(\bar{s}), \qquad p_{N+1}(\bar{s}) = [a_\alpha\bar{s} + b_\alpha]q_y^\alpha(\bar{s}) \\
&p_i(\bar{s}) = [a_\alpha\bar{s} + b_\alpha]\overset{'}{q}{}_i^\alpha(\bar{s}), \qquad \tilde{p}_{N+1}(\bar{s}) = [a_\alpha\bar{s} + b_\alpha]\overset{'}{q}{}_y^\alpha(\bar{s})
\end{aligned} \tag{23}$$

(where $\hat{p} = p+2(N+1)$) for any constants a_α, b_α at any point $\bar{s}$ in $[t_o^o\,,\, t_o^1]$.

Next, set

$$\begin{aligned}
\lambda_\rho &= \bar{\lambda}_\rho & \rho &= 0,1,\ldots,p+2N+1 \\
\lambda_{\bar{\bar{p}}+\tau} &= \bar{\lambda}_{\bar{\bar{p}}+1+\tau} & \tau &= 1,\ldots,2m; \quad \bar{\bar{p}} = p + 2N + 1.
\end{aligned} \tag{24}$$

Then by further application of Theorem 3.2 of [2] together with our selection of the arc a_o, we see that the statement involving (10) together with all of the properties of the multipliers $\mu_\alpha(s)$ as listed in Theorem 3.1 are true.

In order to establish those results of Theorem 3.1, not yet proven, we note by the form of $\bar{H}$ together with the facts that $\nu\equiv 0$, $\overset{'}{y}\equiv 1$ along a_o that there is a function H as described above Theorem 3.1 such that along $\bar{a}_o$, a_o we have

$$\bar{H}_{\overset{'}{x}{}^i} = H_{\dot{x}^i} \qquad \bar{H}_{x^i} = H_{x^i} \qquad \bar{H}_{p_i} = H_{p_i} \qquad \bar{H}_{u^k} = H_{u^k} \qquad \bar{H}_y = H_t\,. \tag{25}$$

Thus, by (21) and the selection of the arc $\bar{a}_o$ we see that the statement involving (7) is proven. Next, by the last relation of (21) we obtain

$$0 = \bar{H}_\nu = p_i \frac{x'^i}{y'} + p_{N+1} - \mu_\alpha \tilde{\phi}^\alpha \tag{26-1}$$

which yields

$$p_{N+1} = \mu_\alpha \tilde{\phi}^\alpha - p_i \frac{x'^i}{y'} \tag{26-2}$$

and by differentiating[(4)]

$$p'_{N+1} = \mu'_\alpha \tilde{\phi}^\alpha + \mu_\alpha \phi^{\alpha'} y' - p_i f^i - p'_i \frac{x'^i}{y'} \quad . \tag{27}$$

Next, by the second relation of (21) we get that on intervals of continuity of $\bar{\mu}_o(s)$

$$p'_{N+1} = -\bar{H}'_y = \bar{H}_{x'^i} x'^i - 2p_i f^i - \tilde{p}_i x'^i - \tilde{p}_{N+1}$$

$$+ \bar{\lambda}_o L_o + \bar{\lambda}_\gamma L_\gamma + 2\mu_\alpha \phi^\alpha + \tilde{\mu}_\alpha \tilde{\phi}^\alpha \qquad \gamma=1,\ldots,p \; . \tag{28}$$

Then equating (28) with (27), solving for $\tilde{p}_{N+1}$ using the first relation of (21), together with the fact that $y' \equiv 1$ along $\bar{a}_o$, we have on $\bar{a}_o$

$$\tilde{p}_{N+1} = -[p_i f^i + \tilde{p}_i x'^i - \bar{\lambda}_o L_o - \bar{\lambda}_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}_\alpha \tilde{\phi}^\alpha + \mu'_\alpha \tilde{\phi}^\alpha] \; . \qquad \gamma=1,\ldots,p \tag{29}$$

Thus on $\bar{a}_o$

$$\tilde{p}_{N+1} = -H - \mu'_\alpha \tilde{\phi}^\alpha \quad . \tag{30}$$

Now by the properties of the multipliers $\mu_\alpha(s)$ and by reasoning similar to that used in (26-2) we have that

$$\mu''_\alpha(s)\tilde{\phi}^\alpha(s) = 0 \qquad 1 \le \alpha \le m \qquad (\alpha \text{ not summed}). \tag{31}$$

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(4) By the properties of $\mu_\alpha(s)$, then for each $\alpha, \mu'_\alpha(s)$ exists if $\psi^\alpha(s) < 0$. Thus $\mu'_\alpha(s)\tilde{\phi}^\alpha(s)$ (α not summed) exists if $\psi^\alpha(s) < 0$. If $\psi^\alpha(s) = 0$ then $\tilde{\phi}^\alpha(s) = 0$ and we define $\mu_\alpha(s)\tilde{\phi}^\alpha(s) = 0$ (α not summed).

Then by differentiating (30) and using (31) together with the fourth relation of (21), the last relation of (25) and the selection of the arc a_o we have

$$-H_t = \tilde{p}'_{N+1} = -[\dot{H} + \dot{\mu}_\alpha \phi^\alpha] \tag{32}$$

on intervals of continuity of $\bar{u}_o(s)$, proving (8). The statement preceeding (8) follows from (30) together with the properties of $\tilde{p}_{N+1}(s)$ and $\mu'_\alpha(s)\tilde{\phi}^\alpha(s)$. Next by (24), (30) the definition of $\bar{G}$, T'^s, the values of $\bar{b}_o^{r+s}$ and the selection of $\bar{a}_o$ we see that the part of (22) which depends upon $b',\ldots,b^r$ may be written as (9), thus proving the statement involving that relation.

By our selection of $\bar{a}_o$, together with the definition of H, $\bar{H}$, we see that at points $\pi = (t, x_o(t), \dot{x}_o(t), u)$ in R and $\Pi = (s, \bar{x}_o(s), \bar{x}'_o(s), \bar{y}_o(s), \bar{y}'_o(s), u, \nu)$ in $\bar{R}$ with $s = \bar{y}_o(s) = t$, $\bar{x}_o(s) = x_o(t)$, $\bar{x}'_o(s) = \dot{x}_o(t)$, $\bar{y}'_o(s) = 1$, $\nu = 0$, then

$$\bar{H} = H + \tilde{p}_{N+1} \tag{33}$$

where $\bar{H}, H$ are evaluated respectively at Π and π. Furthermore given any point π in R_o then we can find a point Π in $\bar{R}_o$ related to π as described above. Then by (33) and (20) we see that (6) is proven.

Now multipliers of the form specified in (11) imply by (26-2) and the selection of $\bar{a}_o$, that

$$p_{N+1}(\bar{t}) = [a_\alpha \bar{t} + b_\alpha]\left[\tilde{\phi}^\alpha(\bar{t}) - q_i^\alpha(\bar{t})\frac{\bar{x}'^i_o(\bar{t})}{\bar{y}'_o(\bar{t})}\right] = (a_\alpha \bar{t} + b_\alpha) q_t^\alpha(\bar{t}) \tag{34}$$

and furthermore by using (30) and the definition of ϕ^α

$$\tilde{p}_{N+1}(\bar{t}) = -[a_\alpha \bar{t} + b_\alpha][q_i^\alpha(\bar{t}) f^i + q'^\alpha_i(\bar{t}) x'^i - \phi^\alpha] = (a_\alpha \bar{t} + b_\alpha) q'^\alpha_t(\bar{t}) \tag{35}$$

so that the situation described in (23) exists. Thus (11) and also Theorem 3.1 are proven.

It is noted that Theorem 3.1 was proven without the aid of (19) and its associated remarks from Theorem 3.2 of [2]. Now with $T^s(b) = t_o^s \quad s=0,1$ identically for all b we see that the problem Section 2 of [2] is of the form described in Section 2 of this paper. All of the results of Theorem 3.1 apply and establish the results of Theorem 3.2 of [2] with (19) and its associated remarks following from the corresponding items of Theorem 3.1. Thus the proofs of Theorem 3.2 of [2] and hence also Theorem 3.1 of [2] are complete.

6. A Problem with Constraints Involving $u, x, \dot{x}$ and Equality State Constraints.

As the next step in our generalization process consider the problem of Section 2 to which we adjoin constraints of the type

$$\begin{aligned} &\psi(t,x,) = 0 \\ &\theta(t,x,\dot{x},u) = 0 \,. \end{aligned} \tag{36}$$

More precisely, let the regions R, B, the functions ψ^α, $\tilde{\phi}^\alpha$, ϕ^α, $\alpha = 1,\dots,m'$, f^i, L_γ, X^{is}, $\dot{X}^{is}$, T^s, g_γ be as described there. For present purposes we desire to minimize the integral

$$I_o(a) = g_o(b) + \int_{t_o}^{t^1} L_o(t,x(t),\dot{x}(t),u(t))dt \tag{37}$$

on the class C of arcs which have points $(t,x(t),\dot{x}(t),u(t))$ in R, b in B, with $u(t)$ piecewise continuous, $x(t)$ of class C^1 and which satisfy the conditions:

$$\ddot{x}^i = f^i(t,x(t),\dot{x}(t),u(t)) \qquad 1 \le i \le N \tag{38-1}$$

$$\psi^\alpha(t,x(t)) \le 0 \quad 1 \le \alpha \le m' \,, \quad \psi^\alpha(t,x(t)) = 0 \quad m' < \alpha \le m \tag{38-2}$$

$$\theta^\eta(t,x(t),\dot{x}(t),u(t)) = 0 \qquad 1 \le \eta \le L \tag{38-3}$$

$$I_\gamma(a) \le 0 \quad 1 \le \gamma \le p' \,, \quad I_\gamma(a) = 0 \quad p' < \gamma \le p \tag{38-4}$$

$$x^i(t^s) = X^{is}(b) \qquad \dot{x}^i(t^s) = \dot{X}^{is}(b) \qquad 1 \leq i \leq N \qquad (3$$

$$t^s = T^s(b) \qquad s = 0,1$$

where:

$$I_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} L_\gamma(t,x(t),\dot{x}(t),u(t))dt \qquad \gamma = 1,\ldots,p \quad .$$

The functions θ^η $\eta=1,\ldots,L$, ψ^α $\alpha=m' + 1,\ldots,m$ will be assumed to be respectively of class C^1 , C^3 on R while the other functions are assumed to have the continuity properties described in Section 2.

Let R_o be the set of points in R satisfying

$$\psi^\alpha \leq 0 \quad 1 \leq \alpha \leq m' \qquad \psi^\alpha = 0 \qquad m' < \alpha \leq m \qquad (3$$

$$\theta^\eta = 0 \qquad 1 \leq \eta \leq L \qquad (3$$

$$\tilde{\phi}^\alpha = 0 \text{ and } \phi^\alpha \geq 0 \quad \forall\alpha \ni \psi^\alpha = 0 \text{ or } \tilde{\phi}^\alpha = 0 \text{ and } \phi \leq 0 \quad \forall\alpha \ni \psi^\alpha = 0 \quad 1 \leq \alpha \leq m' \quad (3$$

$$\tilde{\phi}^\alpha = 0 \qquad \phi^\alpha = 0 \qquad m' < \alpha \leq m \qquad (3$$

where also for $m' < \alpha \leq m$ we have defined $\tilde{\phi}^\alpha = d\psi^\alpha/dt$ and $\phi^\alpha = d^2\psi^\alpha/dt^2$ along arcs satisfying (38-1).

We wish to test whether the arc

$$a_o: \qquad x_o(t), \qquad \dot{x}_o(t), \qquad u_o(t), \qquad b_o \qquad t_o^o \leq t \leq t_o^1$$

which is in the class C is a solution to the above problem. Analogous to Section 2, we define the set R_1 as the collection of points $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. It will be assumed that the matrix

$$\begin{bmatrix} \theta^\eta_{u^k} & 0 \\ \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \eta=1,\ldots,L; \quad \alpha,\beta=1,\ldots,m; \quad k=1,\ldots,K \qquad (4)$$

has rank $L + m$ on $(R_1)^-$, the closure of R_1 in R. Define the functions

$$H(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu},h) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}_\alpha \tilde{\phi}^\alpha - h_\eta \theta^\eta \tag{41-1}$$

$$\begin{aligned} G(b) = \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+i} X^{io} + \lambda_{p+N+1} T^o + \lambda_{\bar{p}+i} \dot{X}^{io} - \lambda_{\bar{\bar{p}}+\alpha} (\dot{q}_i^\alpha(t_o^1) X^{il} \\ + \dot{q}_t^\alpha(t_o^1) T^1 + q_i^\alpha(t_o^1) \dot{X}^{il}) + \lambda_{\bar{\bar{p}}+m+\alpha} (\dot{q}_i^\alpha(t_o) X^{io} + \dot{q}_t^\alpha(t_o^o) T^o + q_i^\alpha(t_o^o) \dot{X}^{io}) \end{aligned} \tag{41-2}$$

$i = 1,\ldots,N$; $\gamma = 1,\ldots,p$; $\alpha = 1,\ldots,m$; $\eta = 1,\ldots,L$; $\bar{p}=p+N+1$; $\bar{\bar{p}}=p+2N+1$.

It will be shown that an arc a_o as described above must satisfy the conditions of the following theorem in order to be a solution to the present problem.

Theorem 6.1 Suppose that the arc a_o is a solution to the above problem. Then there are multipliers

$$K^\tau, \lambda_\rho, \mu_\alpha(t), \tilde{\mu}_\alpha, h_\eta(t), p_i(t), \tilde{p}_i(t) \qquad t_o^o \leq t \leq t_o^1 \tag{42}$$

$\tau = 1,\ldots,2m$; $\rho = 0,1,\ldots,p+2N+1+2m$; $\alpha = 1,\ldots,m$; $\eta = 1,\ldots,L$; $i = 1,\ldots,N$

and functions H and G as described above such that with these multipliers as arguments, then the following conditions hold: The inequality

$$\begin{aligned} H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t)) \leq \\ H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t)) \end{aligned} \tag{43}$$

is valid for all u with $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. The multipliers $p_i(t),\tilde{p}_i(t)$ are each continuous with a piecewise continuous derivative on $[t_o^o, t_o^1]$. Together with $\mu(t),\tilde{\mu},h(t),\lambda$, they satisy the relations

$$\dot{p}_i = -H_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -H_{x^i} \qquad \ddot{X}^i = H_{p_i} \qquad H_{u^k} = 0 \tag{44}$$

along a_o on intervals of continuity of $u_o(t)$. The function $H(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t))$ is continuous along a_o.

On intervals of continuity of $u_o(t)$, this function has a continuous derivative satisfying

$$dH/dt + \dot{\mu}_\alpha \phi^\alpha = H_t \quad . \tag{45}$$

The transversality relation

$$dG + \Big[[-H(t_o^s) - \dot{\mu}_\alpha(t_o^s)\tilde{\phi}^\alpha(t_o^s)]dT^s + p_i(t_o^s)d\dot{X}^{is} + \tilde{p}_i(t_o^s)dX^{is} \Big]_{s=0}^{s=1} = 0$$

is valid along a_o for all db.

The multipliers λ_ρ, K^τ are constants which satisfy:

$$\lambda_o \geq 0 \qquad \lambda_\gamma \geq 0 \text{ with } \lambda_\gamma = 0 \quad \text{if} \quad I_\gamma(a_o) < 0 \qquad 1 \leq \gamma \leq p'$$

$$K^\alpha \geq 0 \quad 1 \leq \alpha \leq m' \qquad \lambda_{p+i} = K^\alpha q_i^\alpha(t_o^o) + K^{m+\alpha}\, \dot{q}_i^\alpha(t_o^o) \tag{47}$$

$$\lambda_{p+N+1} = K^\alpha q_t^\alpha(t_o^o) + K^{m+\alpha} \dot{q}_t^\alpha(t_o^o) \qquad \lambda_{p+N+1+i} = K^{m+\alpha} q_i^\alpha(t_o^o) \qquad 1 \leq \alpha \leq m; \quad 1 \leq i \leq N \; .$$

Furthermore, together with $\mu(t), \tilde{\mu}, p(t), \tilde{p}(t)$ they are not of the form

$$\lambda_\gamma = 0 \qquad \gamma = 0,1,\ldots,p; \qquad K^\tau = 0 \qquad \tau = 1,\ldots,2m \tag{48}$$

$$\dot{\mu}_\alpha(\bar{t}) = a_\alpha \quad \text{if} \quad \psi^\alpha(\bar{t}) < 0 \qquad \mu_\alpha(\bar{t}) = a_\alpha \bar{t} + b_\alpha \qquad \tilde{\mu}_\alpha = a_\alpha \qquad \lambda_{\bar{\bar{p}}+\alpha} = a_\alpha t_o^1 + b_\alpha$$

$$\lambda_{\bar{\bar{p}}+m+\alpha} = a_\alpha t_o^o + b_\alpha \qquad p_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha) q_i^\alpha(\bar{t}) \qquad \tilde{p}_i(\bar{t}) = (a_\alpha \bar{t} + b_\alpha)\dot{q}_i^\alpha(\bar{t})$$

$$\bar{\bar{p}} + p + 2N + 1; \qquad 1 \leq i \leq N; \qquad 1 \leq \alpha \leq m$$

for any constants a_α , b_α at any point $\bar{t}$ in $[t_o^o , t_o^1]$.

The multipliers $\mu_\alpha(t)$, $h_\eta(t)$, are continuous on intervals of continuity of $u_o(t)$. Furthermore, for each α, $1 \leq \alpha \leq m'$, the multipliers $\mu_\alpha(t)$ satisfy the following properties: (i) there are constants a_α , b_α such that $\mu_\alpha(t) - (a_\alpha t + b_\alpha)$ is a nonincreasing function on $[t_o^o , t_o^1]$; (ii) it is of the form $\bar{a}_\alpha t + \bar{b}_\alpha$ on intervals upon which $\psi^\alpha(t) < 0$ and (iii) $\dot{\mu}_\alpha(t_o^1) = \tilde{\mu}_\alpha$ if $\psi^\alpha(t_o^1$

7. Auxiliary Lemmas.

Using the notation introduced above, set

$$\theta^{L+\tau} = \phi^{m'+\tau} \qquad 1 \leq \tau \leq m - m' . \tag{49}$$

Next, note that according to the definition of R_1 we have $\psi^\beta = 0$ $\beta = m' + 1,\ldots,m$ on R_1 and hence also on $(R_1)^-$. Thus the rank of the matrix (40) remains intact if the terms $\delta_{\alpha\beta}\psi^\beta$ in the last $m-m'$ rows of (40) are replaced by zero. Hence the matrix

$$\begin{bmatrix} \theta^\rho_{u^k} & 0 & 0 \\ \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta & 0 \end{bmatrix} = \begin{bmatrix} \theta^\eta_{u^k} & 0 & 0 \\ \phi^\tau_{u^k} & 0 & 0 \\ \phi^\alpha_{u^k} & \delta_{\alpha\beta}\psi^\beta & 0 \end{bmatrix} \quad \begin{matrix} 1\leq\eta\leq L \quad 1\leq\alpha,\beta\leq m' \quad m'<\tau\leq m \\ 1\leq\rho\leq L+m-m' \quad 1\leq k\leq K \end{matrix} \tag{50}$$

(where we have used the identification (49) in (50)) has rank $L + m$ on $(R_1)^-$.

The following result will be used in proving Theorem 6.1.

Lemma 7.1. There exist K real valued functions $U^k(t,x,\dot{x},u)$ $k=1,\ldots,K$ of class C^1 on a neighborhood $N_1 \subseteq R$ of $(R_1)^-$ such that $(t,x,\dot{x},U(t,x,\dot{x},u))$ is in R for $(t,x,\dot{x},u)$ in N_1 and

$$U^k(t,x,\dot{x},u) = u^k \qquad (t,x,\dot{x},u) \text{ in } (R_1)^- \tag{51-1}$$

the matrix $[\partial U^k/\partial u^h(t,x,\dot{x},u)]$ has rank $K-(L + m-m')$ on N_1 (51-2)

$$\theta^\eta(t,x,\dot{x},U(t,x,\dot{x},u)) = 0 \qquad \eta=1,\ldots,L + m-m' \quad \text{on } N_1 \tag{51-3}$$

Proof: With $(R_1)^-$ N_1, B^k_ρ, θ^η, b_ρ replacing respectively R_o, R_1, A^k_β, ϕ^α, B_β of Lemma 3.1. Chapter 5 of [1] and with $p=1$ there, then by the properties of the first $L + m-m'$ rows of the matrix of (50), all of the details of the proof of that lemma apply here to establish the functions

$$U^k(t,x,\dot{x},u) = u^k + B^k_\rho(t,x,\dot{x},u)b_\rho(t,x,\dot{x},u) \tag{52}$$

$$(t,x,\dot{x},u) \text{ in } N_1, \quad k=1,\ldots,K, \quad \rho = 1,\ldots L+m'-m$$

(where B^k_p, b_p are functions of class C^1 on N_1 with $b_\rho = 0$ on $(R_1)^-$) which satisfy all of the conditions of the present lemma except for (51-2). That property may be established by: (i) differentiating (51-3) with respect to u^k on N_1 to get

$$\begin{gathered}\theta^\eta_{u^h}(t,x,\dot{x},U(t,x,\dot{x},u))[\partial U^h/\partial u^k(t,x,\dot{x},u)] = 0 \\ \eta=1,\ldots,L + m-m'; \quad k,h, = 1,\ldots,K\end{gathered} \tag{53}$$

(ii) using the properties of the first $L + m-m'$ rows of the matrix of (50) together with the continuity properties of the functions involved and (iii) noting the form (52).

With $U(t,x,\dot{x},u)$ and N_1 as the objects of Lemma 7.1 and with

$$Z \equiv \{(t,x,\dot{x},u)\ \varepsilon N_1 \,|\, (t,x,\dot{x},U(t,x,\dot{x},u))\varepsilon R_1\}$$

we prove:

<u>Lemma 7.2</u> The matrix

$$\begin{bmatrix} \phi^\alpha_{u^h}\partial U^h/\partial u^k, & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \alpha,\beta=1,\ldots,m' \qquad k,h=1,\ldots,K$$

with the arguments of $\phi^\alpha_{u^k}$ being $(t,x,\dot{x},U(t,x,\dot{x},u))$ and those of $\partial U^h/\partial u^k$ and ψ^β being $(t,x,\dot{x},u)$ and with $\delta_{\alpha\beta}$ as the Kronecker delta, has rank m' for $(t,x,\dot{x},u)$ in Z.

Proof: Fix a point $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ in Z. Note that the matrix (54) is the following matrix product

$$\begin{bmatrix} \phi^\alpha_{u^h}, & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \begin{bmatrix} \dfrac{\partial U^h}{\partial u^k} & 0 \\ 0 & \delta_{\alpha\beta} \end{bmatrix} \qquad \alpha,\beta = 1,\ldots,m' \qquad h,k = 1,\ldots,K \tag{55}$$

where the argument of the first matrix is $(\bar{t},\bar{x},\bar{\dot{x}},U(\bar{t},\bar{x},\bar{\dot{x}},\bar{u}))$ in R_1 and that of the second matrix is $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ in Z. Next consider the vectors $C^1,\ldots,C^{K+m}$

as the $K+m$ dimensional column vectors of

$$\left[C^1,\ldots,C^{K+m}\right] = \begin{bmatrix} \dfrac{\partial U^h}{\partial u^k} & 0 \\ & \\ 0 & \delta_{\alpha\beta} \end{bmatrix} \qquad h,k = 1,\ldots,K \quad \alpha,\beta = 1,\ldots,m \tag{56}$$

where these terms are formed at $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ and let $D^1,\ldots,D^{L+m-m'}$ be the first $L+m-m'$ row vectors of the matrix in (50) and $A^1,\ldots,A^{m'}$ the last m' rows where this latter matrix is evaluated at $(\bar{t},\bar{x},\bar{\dot{x}},U(\bar{t},\bar{x},\bar{\dot{x}},\bar{u}))$ in R_1. According to Lemma 7.1 the rank of $C^1,\ldots,C^{K+m}$ is $K+m-(L+m-m')$.

Now assume that there exists a nontrivial relationship among the rows of the matrix of (54). By the form of (55) this would mean that for some linear combination $V = \rho_i A^i \quad 1 \le i \le m'$ we would have

$$V \cdot C^h = 0 \qquad h = 1,\ldots,K+m\ . \tag{57}$$

Thus V would belong to the orthogonal complement $C^\perp$ of the space C spanned by $C^1,\ldots,C^{K+m}$. However, we know that $C^\perp$ has dimension $L+m-m'$ and that by (53) evaluated at $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ the vectors $D^1,\ldots,D^{L+m-m'}$ span this space. Then we must have that V is a linear combination of $D^1,\ldots,D^{L+m-m'}$ contradicting the rank of (50). Thus the matrix of (54) has rank m' at $(\bar{t},\bar{x},\bar{\dot{x}},\bar{u})$ and hence in all of Z, proving the lemma.

8. Proof of Theorem 6.1

In order to prove Theorem 6.1 we use the functions $U(t,x,\dot{x},u)$ of Lemma 7.1 to define the functions

$$\bar{f}^i(t,x,\dot{x},u) = f^i(t,x,\dot{x},U(t,x,\dot{x},u)) \qquad \bar{L}_\gamma(t,x,\dot{x},u) = L_\gamma(t,x,\dot{x},U(t,x,\dot{x},u))$$
$$\bar{\phi}^\alpha(t,x,\dot{x},u) = \phi^\alpha(t,x,\dot{x},U(t,x,\dot{x},u)) \tag{58}$$

on the set N_1 of Lemma 7.1. By the properties of the functions involved, we see that $\bar{f}^i$, $\bar{L}_\gamma$, $\bar{\phi}^\alpha$ are C^1 on N_1. We are now interested in arcs with

points $(t,x(t),\dot{x}(t),u(t))$ in N_1, b in B, $u(t)$ piecewise continuous and $x(t)$ of class C^1.

Consider the following problem: It is desired to minimize the integral

$$\bar{I}_o(a) = g_o(b) + \int_{t^o}^{t^1} \bar{L}_o(t,x(t),\dot{x}(t),u(t))dt \tag{59}$$

on the class $\bar{C}$ of arcs which satisfy the conditions

$$\ddot{x}^i = \bar{f}^i(t,x,\dot{x},u) \qquad 1 \leq i \leq N \tag{60-1}$$

$$\psi^\alpha(t,x) \leq 0 \qquad 1 \leq \alpha \leq m' \tag{60-2}$$

$$\bar{I}_\gamma(a) \leq 0 \quad 1 \leq \gamma \leq p' \qquad \bar{I}_\gamma(a) = 0 \quad p' < \gamma \leq p+2(m-m') \tag{60-3}$$

$$x^i(t^s) = X^{is}(b),\ \dot{x}^i(t^s) = \dot{X}^{is}(b) \quad 1\leq i\leq N \quad t^s=T^s(b) \quad s=0,1 \tag{60-4}$$

where:

$$\bar{I}_\gamma(a) = g_\gamma(b) + \int_{t^o}^{t^1} \bar{L}_\gamma(t,x(t),\dot{x}(t),u(t))dt \qquad \gamma=1,\dots,p$$

$$\bar{I}_{p+\tau}(a) = \psi^{m'+\tau}(T^o(b),\ X^o(b)) = 0$$

$$\bar{I}_{p+m-m'+\tau}(a) = \tilde{\phi}^{m'+\tau}(T^o(b),X^o(b),\dot{X}^o(b)) = 0 \qquad 1 \leq \tau \leq m-m'$$

Next, let $\bar{a}$:

$$\bar{a}: \qquad \bar{x}(t), \qquad \dot{\bar{x}}(t), \qquad \bar{u}(t), \qquad \bar{b} \qquad t^o \leq t \leq t^1$$

be an arc in $\bar{C}$. The arc

$$a: \quad x(t) = \bar{x}(t)\ , \quad \dot{x}(t) = \dot{\bar{x}}(t)\ , \quad u(t) = U(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t)),\ b=\bar{b} \quad t^o\leq t\leq t^1$$

is in R and by Lemma 7.1 together with the definition of the functions $\tilde{\phi}^\tau$, ϕ^τ $\tau=m'+1,\dots,m$, satisfies the conditions (38) and hence is in the class C. Moreover, with a_o as the arc introduced below (39) we have by (51-1) that

$$U(t,x_o(t),\dot{x}_o(t),u_o(t)) = u_o(t)\ . \tag{61}$$

Then as a_o satisfies the conditions (38) it also satisfies the conditions (60) and is in the class $\bar{C}$. Finally for arcs a and $\bar{a}$ constructed as above, we have

$$I_\gamma(a) = \bar{I}_\gamma(a) \qquad \gamma = 0,1,\dots,p\,. \tag{62}$$

Thus if a_o minimizes I_o on the class C, then it also minimizes $\bar{I}_o$ on the class $\bar{C}$.

Define N_2 as the collection of points $(t,x,\dot{x},u)$ in N_1 satisfying

$$\psi^\alpha \le 0$$

$$\tilde{\phi}^\alpha=0 \text{ and } \bar{\phi}^\alpha \ge 0 \text{ for } \forall_\alpha \text{ with } \psi^\alpha=0 \quad\text{or}\quad \tilde{\phi}^\alpha=0 \text{ and } \bar{\phi}^\alpha \le 0 \text{ for } \forall_\alpha \text{ with } \psi^\alpha=0 \quad 1\le\alpha\le m' \,.$$

In addition, define N_3 as the set of points $(t,x_o(t),\dot{x}_o(t),u)$ in N_2. Let $(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u})$ be a point in N_3. Then by the definition of the functions $\bar{\phi}^\alpha$ and by Lemma 7.1 we see that the point $(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),U(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u}))$ satisfies (39) and hence is in R_1. Thus by Lemma 7.2 we see that the matrix

$$\begin{bmatrix} \bar{\phi}^\alpha_{u^k}(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),\bar{u}) & \delta_{\alpha\beta}\psi^\beta(\bar{t},x_o(\bar{t})) \end{bmatrix} \qquad 1 \le \alpha,\ \beta \le m' \tag{64}$$

has rank m' on N_3.

Set

$$\begin{aligned}\bar{G}(b) = {}& \lambda_o g_o + \lambda_\gamma g_\gamma + \lambda_{p+\tau}\psi^{m'+\tau}(T^o,X^o) + \lambda_{p+m-m'+\tau}\tilde{\phi}^{m'+\tau}(T^o,X^o,\dot{X}^o) \\ &+ \lambda_{\hat{p}+i}X^{io} + \lambda_{\hat{p}+N+1}T^o + \lambda_{\bar{p}+i}\dot{X}^{io} - \lambda_{\bar{p}+N+\alpha}\left(\dot{q}^\alpha_i(t^1_o)X^{i1} + \dot{q}^\alpha_t(t^1_o)T^1 + q^\alpha_i(t^1_o)\dot{X}^{i1}\right) \\ &+ \lambda_{\bar{p}+N+m'+\alpha}\left(\dot{q}^\alpha_i(t^o_o)X^{io} + \dot{q}^\alpha_t(t^o_o)T^o + q^\alpha_i(t^o_o)\dot{X}^{io}\right)\end{aligned} \tag{65}$$

and

$$\bar{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu})=p_i\bar{f}^i + \tilde{p}_i\dot{x}^i - \lambda_o\bar{L}_o - \lambda_\gamma\bar{L}_\gamma - \mu_\alpha\bar{\phi}^\alpha - \tilde{\mu}_\alpha\tilde{\phi}^\alpha \tag{66}$$

$$\hat{p}=p+2(m-m'); \quad \bar{p}=\hat{p}+N+1; \quad \tau=1,\dots,m-m'; \quad 1\le i\le N; \quad 1\le\gamma\le p; \quad 1\le\alpha\le m' \,.$$

By the above argument, we see that Theorem 3.1 applies to the problem introduced in this section. Thus there are functions $\bar{H}$, $\bar{G}$ as described above and multipli[ers] $\bar{K}^{\tau}$, $\bar{\lambda}_{\rho}$, $p_i(t), \tilde{p}_i(t), \mu_{\alpha}(t), \tilde{\mu}_{\alpha}$ $\quad \tau=1,\ldots,2m'$, $\rho = 0,1,\ldots,p+2(m-m') + 2N + 1 + 2m'$, $i=1,\ldots,N$, $\alpha=1,\ldots,m'$ such that with these multipliers as arguments, then the following conditions hold: The relation

$$d\bar{G} + \Big[-(\bar{H}(t_o^s) + \dot{\mu}_{\alpha}(t_o^s)\tilde{\phi}^{\alpha}(t_o^s))dT^s + p_i(t_o^s)d\dot{X}^{is} + \tilde{p}_i(t_o^s)dX^{is}\Big]_{s=0}^{s=1} = 0 \qquad (6$$

is valid along a_o for all db and

$$\bar{H}(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \leq \bar{H}(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \quad (6$$

holds whenever $(t,x_o(t),\dot{x}_o(t),u)$ is in N_3. On intervals of continuity of $u_o(t)$, the relations

$$\dot{p}_i = -\bar{H}_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -\bar{H}_{x^i} \qquad \bar{H}_{u^k} = 0 \qquad \frac{d\bar{H}}{dt} + \dot{\mu}_{\alpha}(t)\phi^{\alpha}(t) = \bar{H}_t \qquad 1 \leq \alpha \leq m' \qquad (6$$

hold. The multipliers $\bar{\lambda}_{\rho}, K^{\alpha}$ satisfy

$$\bar{\lambda}_o \geq 0 \qquad \bar{\lambda}_{\gamma} \geq 0 \text{ with } \bar{\lambda}_{\gamma} = 0 \text{ if } \bar{I}_{\gamma}(a_o) < 0 \qquad 1 \leq \gamma \leq p \qquad \bar{K}^{\alpha} \geq 0 \qquad (7$$

$$\bar{\lambda}_{\hat{p}+i} = \bar{K}^{\alpha}q_i^{\alpha}(t_o^o) + \bar{K}^{m'+\alpha}\dot{q}_i^{\alpha}(t_o^o) \qquad \bar{\lambda}_{\hat{p}+N+1} = \bar{K}^{\alpha}q_t^{\alpha}(t_o^o) + \bar{K}^{m'+\alpha}\dot{q}_t^{\alpha}(t_o^o) \qquad \bar{\lambda}_{\hat{p}+N+1+i} = \bar{K}^{m'+\alpha}q_i^{\alpha}($$

$$\hat{p} = p + 2(m-m') \qquad 1 \leq \alpha \leq m'$$

and are not of the form:

$$\bar{\lambda}_{\gamma} = 0 \qquad \gamma = 0,1,\ldots,p+2(m-m') \qquad \bar{K}^{\tau} = 0 \qquad \tau = 1,\ldots,2m' \qquad (7$$

$$\dot{\mu}_{\alpha}(\bar{t}) = a_{\alpha} \text{ if } \psi^{\alpha}(\bar{t}) < 0 \qquad \mu_{\alpha}(\bar{t}) = a_{\alpha}\bar{t} + b_{\alpha} \qquad \tilde{\mu}_{\alpha} = a_{\alpha} \qquad \bar{\lambda}_{\hat{p}+2N+1+\alpha} = a_{\alpha}t_o^1 +$$

$$\bar{\lambda}_{\hat{p}+2N+1+m'+\alpha} = a_{\alpha}t_o^o + b_{\alpha} \qquad p_i(\bar{t}) = (a_{\alpha}\bar{t}+b_{\alpha})q_i^{\alpha}(\bar{t}) \qquad \tilde{p}_i(\bar{t}) = (a_{\alpha}\bar{t}+b_{\alpha})\dot{q}_i^{\alpha}(\bar{t})$$

$$\hat{p} = p + 2(m-m') \qquad 1 \leq i \leq N \qquad 1 \leq \alpha \leq m'$$

for any constants a_{α}, b_{α} at any point $\bar{t}$ in $[t_o^o, t_o^1]$. Furthermore, the statemen[ts] in Theorem 6.1 involving the properties of the multipliers $\mu_{\alpha}(t)$ $\quad 1 \leq \alpha \leq m'$ follow directly from analogous statements in Theorem 3.1.

In order to prove Theorem 6.1 we use the multipliers introduced above (67) as arguments and define the function

$$\overset{*}{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu}) = p_i f^i + \tilde{p}_i \dot{x}^i - \bar{\lambda}_o L_o - \bar{\lambda}_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}^\alpha \tilde{\phi}^\alpha . \tag{72}$$

Then

$$\bar{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu}) = \overset{*}{H}(t,x,\dot{x},U(t,x,\dot{x},u),p,\tilde{p},\mu,\tilde{\mu}) \qquad (t,x,\dot{x},u) \text{ in } N_1. \tag{73}$$

Next, let $(t,x_o(t),\dot{x}_o(t),u)$ be a point in R_1. Then by (51-1) we see that $\phi^\alpha = \bar{\phi}^\alpha \quad 1 \leq \alpha \leq m'$ at this point so that satisfaction of the conditions (39) implies satisfaction of the conditions (63) and $(t,x_o(t),\dot{x}_o(t),u)$ is in N_2. Thus by (68), (73) and (51-1) we obtain

$$\overset{*}{H}(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \leq \overset{*}{H}(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu}) \tag{74}$$

for $(t,x_o(t),\dot{x}_o(t),u)$ in R_1. Hence at a point $(t,x_o(t),\dot{x}_o(t),u)$ in R_1 we have for all u in a neighborhood of $u_o(t)$ and satisfying

$$\begin{aligned} &\theta^\eta(t,x_o(t),\dot{x}_o(t),u) = 0 && \eta = 1,\ldots,L+m-m' \\ &\phi^\alpha(t,x_o(t),\dot{x}_o(t),u) - \phi^\alpha(t,x_o(t),\dot{x}_o(t),u_o(t))=0 \quad \forall\alpha \ni \psi^\alpha(t)=0 && 1 \leq \alpha \leq m' \end{aligned} \tag{75}$$

that (74) holds (where in 75) we have used the identification $\theta^{L+\tau} = \phi^\tau \quad m'<\tau \leq m$ introduced above Lemma (7.1).

Define the funcrion

$$\tilde{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu},\nu,h) = \overset{*}{H} - h_\eta \theta^\eta - \nu^\alpha \phi^\alpha \qquad 1 \leq \alpha \leq m' ; \quad 1 \leq \eta \leq L+m-m'. \tag{76}$$

By the properties of the matrix of (50) and by using arguments analogous to those used in proving Theorem 4.1 of chapter 5 of [1],there is a unique set of multipliers $\nu^\alpha(t),\ h_\eta(t) \quad \alpha = 1,\ldots,m' \quad \eta = 1,\ldots,L+m-m'$ satisfying

$$\tilde{H}_{u^k}(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t),\nu(t)) = 0 \tag{77-1}$$

and

$$\nu^\alpha(t)\psi^\alpha(t,x_o(t)) = 0 \qquad 1 \leq \alpha \leq m' \qquad \alpha \text{ not summed.} \tag{77-2}$$

In addition, these multipliers are piecewise continuous and are continuous whenever $u_o(t)$ is continuous. Furthermore by (73) and (51-3) we have that

$$H(t,x,\dot{x},U(t,x,\dot{x},u),p,\tilde{p},\mu,\tilde{\mu},h,\nu)+\nu^{\alpha}\phi^{\alpha}(t,x,\dot{x},U(t,x,\dot{x},u)) = \bar{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu})$$

for $(t,x,\dot{x},u)$ in N_1. Differentiating with respect to u at $(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t),\nu(t))$ we obtain

$$\left[\tilde{H}_{u^h} + \nu^{\alpha}\phi^{\alpha}_{u^h}\right]\left[\frac{\partial U^h}{\partial u^k}\right] = \bar{H}_{u^k} \qquad h,k = 1,\ldots,K.$$

Then by (69), (77-1) and (51-1) these results

$$\nu^{\alpha}(t)\phi^{\alpha}_{u^h}(t,x_o(t),\dot{x}_o(t),U(t,x_o(t),\dot{x}_o(t),u_o(t))\ \frac{\partial U^h}{\partial u^k}\ (t,x_o(t),\dot{x}_o(t),u_o(t)) = 0.$$

Now for any point $\bar{t}$ in $[t_o^0, t_o^1]$, let $\alpha_1,\ldots,\alpha_s$ $1 \leq \alpha_i \leq m'$ be the indices for which $\psi^{\alpha}(\bar{t}) = 0$. Then by our selection of the multipliers $\nu^{\alpha}(t)$, the summation in (80) may be taken only over $\alpha_1,\ldots,\alpha_s$. However as $(\bar{t},x_o(\bar{t}),\dot{x}_o(\bar{t}),u_o(\bar{t}))$ is in N_3 then by the remarks involving (64), we see that (80) implies that

$$\nu^{\alpha_1}(\bar{t}) = ,\ldots, = \nu^{\alpha_s}(\bar{t}) = 0 .$$

Thus the multipliers $\nu^{\alpha}(t)$ $\alpha = 1,\ldots,m'$ vanish identically so that we can allow the function $\tilde{H}$ to have the form

$$\tilde{H} = \overset{*}{H} - h_{\eta}\theta^{\eta} = p_i f^i + \tilde{p}_i\dot{x}^i - \bar{\lambda}_o L_o - \bar{\lambda}_{\gamma} L_{\gamma} - \mu_{\alpha}\phi^{\alpha} - \tilde{\mu}_{\alpha}\tilde{\phi}^{\alpha} - h_{\eta}\theta^{\eta}$$

$$1 \leq i \leq N; \qquad 1 \leq \gamma \leq p; \qquad 1 \leq \alpha \leq m', \qquad 1 \leq \eta \leq L + m - m'$$

and by (74) together with the definition of R_o we then have that

$$\tilde{H}(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t)) \leq \tilde{H}(t,x_o(t),\dot{x}_o(t),u_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},$$

for all points $(t,x_o(t),\dot{x}_o(t),u)$ in R_o.

Next, by (82), (73) and (51-3) these results

$$\tilde{H}(t,x,\dot{x},U(t,x,\dot{x},u),p,\tilde{p},\mu,\tilde{\mu},h) = \bar{H}(t,x,\dot{x},u,p,\tilde{p},\mu,\tilde{\mu}) \qquad (t,x,\dot{x},u) \quad \text{in} \quad N_1. \tag{84}$$

Then with arguments $t, x_o(t), \dot{x}_o(t), u_o(t), p(t), \tilde{p}(t), \mu(t), \tilde{\mu}, h(t)$ we obtain by (77-1) and (51-1)

$$\begin{aligned}
\bar{H}_{\dot{x}^i} &= \tilde{H}_{\dot{x}^i} + \tilde{H}_{u^h} \frac{\partial U^h}{\partial \dot{x}^i} = \tilde{H}_{\dot{x}^i} \\
\bar{H}_{x^i} &= \tilde{H}_{x^i} + \tilde{H}_{u^h} \frac{\partial U^h}{\partial x^i} = \tilde{H}_{x^i} \\
\bar{H}_t &= \tilde{H}_t + \tilde{H}_{u^h} \frac{\partial U^h}{\partial t} = \tilde{H}_t \\
\dot{\bar{H}} &= \dot{\tilde{H}}
\end{aligned} \tag{85}$$

so that by (67), (69), (77-1), (84) and the fact that $\phi^{m'+\tau} = \tilde{\phi}^{m'+\tau} = 0$ on a_o $\tau = 1,\ldots,m-m'$ we have

$$\dot{p}_i = -\tilde{H}_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -\tilde{H}_{x^i} \qquad \tilde{H}_{u^k} = 0 \qquad \dot{\tilde{H}} + \dot{\mu}_\alpha(t)\phi^\alpha(t) + \dot{h}_{L+\tau}(t)\phi^{m'+\tau}(t) = \tilde{H}_t \tag{86}$$

$$d\bar{G} + [-(\tilde{H}(t_o^s) + \dot{\mu}_\alpha(t_o^s)\tilde{\phi}^\alpha(t_o^s) + \dot{h}_{L+\tau}(t_o^s)\tilde{\phi}^{m'+\tau}(t_o^s))dT^s + p_i(t_o^s)d\dot{X}^{is} + \tilde{p}_i(t_o^s)dX^{is}]_{s=0}^{s=1} = 0$$

$1 \le i \le N \qquad 1 \le k \le K \qquad 1 \le \alpha \le m' \qquad 1 \le \tau \le m - m'$

(where the terms $\dot{h}_{L+\tau}(t)\phi^{m'+\tau}(t)$ and $\dot{h}_{L+\tau}(t)\tilde{\phi}^{m'+\tau}(t)$ are defined to be zero and are included for notational convenience) along a_o on intervals of continuity of $u_o(t)$.

Next, let $a'_{m+\tau}$, $b'_{m+\tau}$ $1 \le \tau \le m-m'$ be arbitrary constants and modify terms in an analogous fashion to that used in (15) of [1]. Thus modify the multipliers $h_{L+t}(t)$ $1 \le \tau \le m-m'$, $\tilde{p}_i(t)$, $p_i(t)$, $1 \le i \le N$ and the functions $\tilde{H}$, $\bar{G}$ to the respective forms:

$$h_{L+\tau}(t) + a'_{m+\tau}t + b'_{m+\tau} \qquad p_i(t) + (a'_{m+\tau}t + b'_{m+\tau})q_i^{m'+\tau}(t)$$

$$\tilde{p}_i(t) + (a'_{m+\tau}t + b'_{m+\tau})\dot{q}_i^{m'+\tau}(t)$$

$$\bar{G} - \bar{\lambda}_{\hat{p}+2N+1+2m'} + \tau\left[\dot{q}_i^{m'+\tau}(t_o^1)X^{i1} + \dot{q}_t^{m'+\tau}(t_o^1)T^1 + q_i^{m'+\tau}(t_o^1)\dot{X}^{i1}\right] \qquad (87$$

$$+ \bar{\lambda}_{\hat{p}+2N+1+2m'+m-m'+\tau}\left[\dot{q}_i^{m'+\tau}(t_o^o)X^{io} + \dot{q}_t^{m'+\tau}(t_o^o)T^o + q_i^{m'+\tau}(t_o^o)\dot{X}^{io}\right]$$

$$\tilde{H} - \tilde{\mu}'_{m+\tau}\tilde{\phi}^{m'+\tau}$$

with: $1 \leq \tau \leq m-m'$; $1 \leq i \leq N$; $\hat{p} = p + 2(m-m')$ and e.g. $q_i^{m'+\tau}(t) = \psi_{x^i}^{m'+\tau}(t)$

with the other terms having analogous meanings and

$$\bar{\lambda}_{\hat{p}+2N+1+2m'+\tau} = a'_{m+\tau}t_o^1 + b'_{m+\tau}$$

$$\bar{\lambda}_{\hat{p}+2N+1+2m'+m-m'+\tau} = a'_{m+\tau}t_o^o + b'_{m+\tau} \qquad (87$$

$$\tilde{\mu}'_{m+\tau} = a'_{m+\tau}$$

and where in $\tilde{H}$, the modified multipliers of (87-1) have been used. Then in a manner analogous to that used in proving Lemma 3.1 of [1] it can be shown that with terms modified as in (87), the statements involving (86), (70), (83), together with the remarks concerning $\mu_\alpha(t)$ $1 \leq \alpha \leq m'$ $h_\eta(t)$ $1 \leq \eta \leq L+m-m'$ hold as before. The statement concerning the prohibited form of the multipliers (71) is replaced by the statement that the multipliers are not for the form:

$$\bar{\lambda}_\gamma=0 \quad \gamma=0,1,\dots,p+2(m-m') \qquad \bar{K}^\delta=0 \quad \delta=1,\dots,2m' \qquad \dot{\mu}_r(\bar{t})=a_r \ \text{ if } \ \psi^r(\bar{t})<0$$

$$\mu_r(\bar{t})=a_r\bar{t}+b_r \qquad \tilde{\mu}_r=a_r \qquad \bar{\lambda}_{\hat{p}+2N+1+r}=a_r t_o^1 + b_r \qquad \bar{\lambda}_{\hat{p}+2N+1+m'+r}=a_r t_o^o + b_r \quad r=1,$$

$$h'_{m+\tau}(\bar{t}) = a'_{m+\tau}\bar{t}+b'_{m+\tau} \qquad \tilde{\mu}'_{m+\tau}=a'_{m+\tau} \qquad \bar{\lambda}_{\hat{p}+2N+1+2m'+\tau}=a'_{m+\tau}t_o^1 + b'_{m+\tau} \qquad (8$$

$$\bar{\lambda}_{\hat{p}+2N+1+2m'+m-m'+\tau} = a'_{m+\tau}t_o^o + b'_{m+\tau} \qquad \tau=1,\dots,m-m' \qquad \hat{p}=p+2(m-m')$$

$$p_i(\bar{t})=(a_\alpha\bar{t} + b_\alpha)q_i^\alpha(\bar{t}) \qquad \tilde{p}_i(\bar{t})=(a_\alpha\bar{t} + b_\alpha)\dot{q}_i^\alpha(\bar{t}) \qquad \alpha=1,\dots,m \qquad i=1,\dots,N$$

for any constants a_α, b_α $\alpha=1,\ldots,m$ at any point $\bar{t}$ in $[t_o^o \; t_o^1]$ (where the additional multipliers $\bar{\lambda}_{\hat{p}+2N+1+2m'+\rho}$ $\rho = 1,\ldots,2(m-m')$ result from modifying $\bar{G}$ in (87-1)).

In order to prove Theorem 6.1, set

$$
\begin{gathered}
K^{\alpha} = \bar{K}^{-\alpha} \qquad K^{m+\alpha} = \bar{K}^{m'+\alpha} \qquad K^{m'+\tau} = \bar{\lambda}_{p+\tau} \qquad K^{m+m'+\tau} = \bar{\lambda}_{p+m-m'+\tau} \qquad \lambda_\gamma = \bar{\lambda}_\gamma \\
\lambda_{p+i} = \bar{\lambda}_{\hat{p}+i} + K^{m'+\tau} q_i^{m'+\tau}(t_o^o) + K^{m+m'+\tau} \dot{q}_i^{m'+\tau}(t_o^o) \qquad \lambda_{p+N+1} = \bar{\lambda}_{\hat{p}+N+1} + K^{m'+\tau} q_t^{m'+\tau}(t_o^o) + K^{m+m'+\tau} \dot{q}_t^{m+m'+\tau}(t_o^o) \\
\lambda_{p+N+1+i} = \bar{\lambda}_{\hat{p}+N+1+i} + K^{m+m'+\tau} q_i^{m'+\tau}(t_o^o) \qquad \lambda_{p+2N+1+\alpha} = \bar{\lambda}_{\hat{p}+2N+1+\alpha} \\
\lambda_{p+2N+1+m+\alpha} = \bar{\lambda}_{\hat{p}+2N+1+m'+\alpha} \qquad \lambda_{p+2N+1+m'+\tau} = \bar{\lambda}_{\hat{p}+2N+1+2m'+\tau} \\
\lambda_{p+2N+1+m+m'+\tau} = \bar{\lambda}_{\hat{p}+2N+1+2m'+m-m'+\tau} \qquad 1 \leq \alpha \leq m', \quad \hat{p} = p+2(m-m') \quad 0 \leq \gamma \leq p \quad 1<i<N \quad 1 \leq \tau \leq m-m'
\end{gathered}
\tag{89}
$$

where $\bar{\lambda}_{\hat{p}+2N+1+2m'+s}$ $1 \leq s \leq 2(m-m')$ are the terms of (87-2).

In addition, set

$$\mu'_{m'+\tau}(t) = h_{L+\tau}(t) \qquad 1 \leq \tau \leq m-m' \quad \text{and} \quad H = \tilde{H} \tag{90}$$

where these terms come from (87). By the above statements together with the definition of $\theta^{L+\tau}$ $1 \leq \tau \leq m-m'$, we see that the function H defined above and the function G defined by using the unbarred multipliers above and the form listed above Theorem 6.1, have the required form of Theorem 6.1. In addition, by the definition of G we have

$$dG = d\bar{G} \tag{91}$$

along a_o for all db. Finally: (i) by the definitions (89), (90) together with the remarks below (87), then the statement involving (43),(44),(45), together with the statements involving the multipliers $\mu_\alpha(t)$ $1 \leq \alpha \leq m$, $h_\eta(t)$ $1 \leq \eta \leq L$ are proven;

(ii) by also using (91), then the statement involving (46) is proven; (iii) by also using (88), the statement involving (48) is proven and (iv) by also using (62) the statement involving (47) is proven thus proving Theorem 6.1.

9. A Problem with Inequality Differential Constraints.

As our final point of generalization we consider next the problem of Section 6 in which we have adjoined constraints of the form

$$\theta(t,x,\dot{x},u) \leq 0 \ .$$

More precisely, let the regions R, B, arcs a, functions ψ^α, $\tilde{\phi}^\alpha$, ϕ^α, θ^η $L' < \eta \leq L$ and f^i, L_γ, X^{is}, $\dot{X}^{is}$, T^s, g_γ be as described there. The problem at hand is to minimize the integral $I_o(a)$ on the class C of arcs with points $(t,x(t),\dot{x}(t),u(t))$ in R, b in B, with $u(t)$ piecewise continuous, $x(t)$ of class C^1 and which in addition satisfy:

$$\ddot{x}^i = f^i(t,x,\dot{x},u) \qquad 1 \leq i \leq N \tag{9}$$

$$\psi^\alpha(t,x) \leq 0 \quad 1 \leq \alpha \leq m' \qquad \psi^\alpha(t,x)=0 \quad m' < \alpha \leq m \tag{9}$$

$$\theta^\eta(t,x,\dot{x},u) \leq 0 \quad 1 \leq \eta \leq L' \qquad \theta^\eta(t,x,\dot{x},u)=0 \quad L' < \eta \leq L \tag{9}$$

$$I_\gamma(a) \leq 0 \quad 1 \leq \gamma \leq p' \qquad I_\gamma(a) = 0 \quad p' < \gamma \leq p \tag{9}$$

$$x^i(t^s)=X^{is}(b) \quad \dot{x}^i(t^s)=\dot{X}^{is}(b) \quad 1\leq i\leq N \quad t^s=T^s(b) \quad s=0,1 \tag{9}$$

where I_o, I_γ are as in Section 6. The adjoined functions θ^η $1 \leq \eta \leq L'$, will be assumed to be of class C^1 on R.

The conditions defining the set R_o now take the form:

$$\psi^\alpha \leq 0 \quad 1 \leq \alpha \leq m' \qquad \psi^\alpha = 0 \quad m' < \alpha \leq m \tag{9}$$

$$\theta^\eta \leq 0 \quad 1 \leq \eta \leq L' \qquad \theta^\eta = 0 \quad L' < \eta \leq L \tag{9}$$

$$\tilde{\phi}^\alpha=0 \quad \phi^\alpha\geq 0 \quad \forall\alpha \ni \psi^\alpha=0 \text{ or } \tilde{\phi}^\alpha=0, \quad \phi^\alpha\leq 0 \quad \forall\alpha \ni \psi^\alpha=0 \quad 1 \leq \alpha \leq m' \tag{9}$$

$$\tilde{\phi}^\alpha = 0 \quad \phi^\alpha = 0 \qquad m' < \alpha \leq m \ . \tag{9}$$

The arc a_o in C and the set R_1 are defined here in a manner analogous to that of Section 6.

Corresponding to (40) we will assume that the matrix

$$\begin{bmatrix} \theta^\eta_{u^k} & \delta_{\eta\rho}\theta^\rho & 0 \\ \phi^\alpha_{u^k} & 0 & \delta_{\alpha\beta}\psi^\beta \end{bmatrix} \qquad \eta,\rho = 1,\ldots,L \quad \alpha,\beta = 1,\ldots,m \tag{94}$$

has rank L+m on $(R_1)^-$, the closure of R_1 in R.

For this problem we will prove the following theorem:

<u>Theorem 9.1</u>. Suppose that the arc a_o is a solution to the above problem. Then with H, G as the functions described above Theorem 6.1 but for the present problem, the arc a_o satisfies the conditions of Theorem 6.1 with (43) replaced by

$$H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu},0) \leq H(t,x_o(t),\dot{x}_o(t),u,p(t),\tilde{p}(t),\mu(t),\tilde{\mu},0) \tag{95}$$

for all u with $(t,x_o(t),\dot{x}_o(t),u)$ in R_o. Moreover the multipliers $h_\eta(t)$ satisfy

$$h_\eta(t)\theta^\eta(t) = 0 \qquad 1 \leq \eta \leq L \qquad \eta \text{ not summed.} \tag{96}$$

and for $1 \leq \eta \leq L'$ are nonnegative functions.

10. <u>Proof of Theorem 9.1.</u>

In order to prove this result, introduce the additional control variables ν^η $\eta = 1,\ldots,L'$ and set

$$\bar{\theta}^\eta = \theta^\eta + (\nu^\eta)^2 \quad 1 \leq \eta \leq L' \qquad \bar{\theta}^\eta = \theta^\eta \quad L' < \eta \leq L \ . \tag{97}$$

Our arcs a now are described as:

a: $\quad x(t), \quad \dot{x}(t), \quad u(t), \quad \nu(t), \quad b \qquad t^0 \leq t \leq t^1$

in t-x-$\dot{x}$-u-ν and b spaces. We shall refer to the constraints (92) with (92-3) replaced by

$$\bar{\theta}^{\eta}(t,x,\dot{x},u,\nu) = 0 \qquad 1 \le \eta \le L \tag{98}$$

and with the other constraints of (92) being unchanged, as the set of modified constraints. We consider as our present problem that one defined by the modified constraints. For this problem $\bar{R}$ is in t-x-$\dot{x}$-u-ν space and the set $\bar{R}_o$ is defined by the conditions:

$$\psi^{\alpha} \le 0 \quad 1 \le \alpha \le m' \qquad \psi^{\alpha} = 0 \quad m' < \alpha \le m \qquad \bar{\theta}^{\eta} = 0 \quad 1 \le \eta \le L \tag{99}$$

$$\tilde{\phi}^{\alpha} = 0 \quad \phi^{\alpha} \ge 0 \ \forall\alpha \ni \ \psi^{\alpha} = 0 \quad \text{or} \quad \tilde{\phi}^{\alpha} = 0 \quad \phi^{\alpha} \le 0 \ \forall\alpha \ni \ \psi^{\alpha} = 0 \quad 1 \le \alpha \le m'$$

$$\tilde{\phi}^{\alpha} = 0 \qquad \phi^{\alpha} = 0 \qquad m' < \alpha \le m .$$

Now set $\nu^{\eta}(t) = \sqrt{-\theta^{\eta}(t)}$ $\eta = 1,\dots,L'$ where this evaluation is done along the arc a_o described above (94) and define the arc $\bar{a}_o$ as

$$\bar{a}_o: \qquad \bar{x}_o(t) = x_o(t) \qquad \dot{\bar{x}}_o(t) = \dot{x}_o(t) \qquad \bar{u}_o(t) = u_o(t)$$

$$\bar{\nu}_o(t) = \nu(t) \qquad t_o^0 \le t \le t_o^1 .$$

Since a_o satisfies the conditions (92) then $\bar{a}_o$ satisfies these conditions as modified above. Furthermore given any arc a satisfying these modified conditions, then the projection of this arc into R will satisfy (92) and hence will be in C. As the value of I_o depends only upon the $t,x,\dot{x},u$ components of points, then we see that $\bar{a}_o$ will be a solution to the present problem if a_o is a solution to the original one.

Let $\bar{R}_1$ be the points $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,\nu)$ in $\bar{R}_o$. By the definition of $\bar{R}_o$ and $\bar{\theta}^{\rho}$, we see that on $\bar{R}_o$

$$\begin{bmatrix} \bar{\theta}^{\rho}_{u^k} & \bar{\theta}^{\rho}_{\nu^{\eta}} & 0 \\ \phi^{\alpha}_{u^k} & \phi^{\alpha}_{\nu^{\eta}} & \delta_{\alpha\beta}\psi^{\beta} \end{bmatrix} = \begin{bmatrix} \theta^{\gamma}_{u^k} & 2\delta_{\gamma\eta}\sqrt{-\theta^{\eta}} & 0 \\ \theta^{\tau}_{u^k} & 0 & 0 \\ \phi^{\alpha}_{u^k} & 0 & \delta_{\alpha\beta}\psi^{\beta} \end{bmatrix} \qquad \begin{array}{l} \rho=1,\dots,L \\ k=1,\dots,K \\ \gamma,\eta=1,\dots,L' \\ \alpha,\beta=1,\dots,m \\ \tau=L'+1,\dots,L \end{array} \tag{100}$$

The matrix on the right-hand side in (100) has arguments in t-x-$\dot{x}$-u space and has rank $L+m$ at a point $(t,x,\dot{x},u)$ in R_o iff the same thing is true of the matrix in (94). Furthermore for a point $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,\nu)$ in $(\bar{R}_1)^-$ the projection $(t,x_o(t),\dot{x}_o(t),u)$ is in $(R_1)^-$. Thus by the assumption involving (94) we see that the matrix on the left-hand side of (100) has rank $L+m$ on $(\bar{R}_1)^-$ the closure of $\bar{R}_1$ in $\bar{R}$.

By the above, we see that the problem modified as above is in the class of problems described in Section 6. Thus there are functions G as described above Theorem 6.1 and $\bar{H}$ where

$$\bar{H}(t,x,\dot{x},u,\nu,p,\tilde{p},\mu,\tilde{\mu},h) = p_i f^i + \tilde{p}_i \dot{x}^i - \lambda_o L_o - \lambda_\gamma L_\gamma - \mu_\alpha \phi^\alpha - \tilde{\mu}^\alpha \tilde{\phi}^\alpha - h_\eta \bar{\theta}^\eta \tag{101}$$

$$1 \le i \le N, \qquad 1 \le \gamma \le p \qquad 1 \le \alpha \le m \qquad 1 \le \eta \le L$$

and multipliers $K^\tau, \lambda_\rho, \mu_\alpha(t), \tilde{\mu}_\alpha, h_\eta(t), p_i(t), \tilde{p}_i(t)$ such that with these multipliers as arguments then the following conditions hold:

The inequality

$$\begin{aligned}\bar{H}(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,\nu,p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t)) \le \\ \bar{H}(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),\bar{u}_o(t),\bar{\nu}_o(t),p(t),\tilde{p}(t),\mu(t),\tilde{\mu},h(t))\end{aligned} \tag{102}$$

is valid for all u,ν with $(t,\bar{x}_o(t),\dot{\bar{x}}_o(t),u,\nu)$ in $\bar{R}_o$. The relations

$$\dot{p}_i = -\bar{H}_{\dot{x}^i} \qquad \dot{\tilde{p}}_i = -\bar{H}_{x^i} \qquad \ddot{x}^i = \bar{H}_{p_i} \qquad \bar{H}_{u^k} = 0 \qquad \bar{H}_{\nu^\eta} = 0 \qquad \dot{\bar{H}} + \dot{\mu}_\alpha \phi^\alpha = \bar{H}_t \tag{103}$$

are satisfied along $\bar{a}_o$ on intervals of continuity of $\bar{u}_o(t)$. Furthermore, the statements involving the continuity properties of $p_i(t)$ and $\tilde{p}_i(t)$, together with (47), (48), and the properties of $\mu_\alpha(t), h_\eta(t)$ hold without revision.

In order to prove the remaining statements of Theorem 9.1 we set

$$H = \bar{H} + h_s(\nu^s)^2 \qquad 1 \le s \le L' \tag{104}$$

with the terms $h_s = h_s(t)$ where these multipliers and the other multipliers of $\bar{H}$ and H are as defined above (102). Then by the definition of the functions $\bar{\theta}^{\eta}$, we see that H has the form required in Theorem 9.1 and also

$$H_x = \bar{H}_x \qquad H_{\dot{x}} = \bar{H}_{\dot{x}} \qquad H_u = \bar{H}_u \qquad H_t = \bar{H}_t \tag{10}$$

Thus by (105) the relations (103) imply (44). Now according to the definition of $\bar{R}_o$ we see that (102) could be written with $h_1(t),\ldots,h_L(t)$ replaced by zeros. Also, given any point $(t,x_o(t),\dot{x}_o(t),u)$ in R_o, we can set $\nu^s = \sqrt{-\theta^s(t,x_o(t),\dot{x}_o(t),u)}$ $\quad 1 \leq s \leq L'$ and the point $(t,x_o(t),\dot{x}_o(t),u,\nu)$ will be in $\bar{R}_o$. Thus with $h_1,\ldots,h_L = 0$ in (102) then (104) and (102) establish (95).

Next, by the fifth relation of (103) we obtain

$$0 = \bar{H}_{\nu^s} = -2h_s(t)\nu_o^s(t) \qquad 1 \leq s \leq L' \qquad s \text{ not summed.} \tag{10}$$

Thus $h_s(t) = 0$ if $\theta^s(t) < 0$ $\quad 1 \leq s \leq L'$ so that (96) holds. The non-negativity of the multipliers $h_s(t)$ $\quad 1 \leq s \leq L'$ follows from (95) together with (106), the properties of the matrix (94) and the Lagrange Multiplier Rule. Finally, by (104) and (106) we have that $H = \bar{H}$ along $\bar{a}_o$, so that by the last relation of (105), then (45) and (46) hold and Theorem 9.1 is proven.

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